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Hyperelliptic loop solitons with genus g: investigations of a quantized elastica

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Abstract

In the previous work [J. Geom. Phys. 39 (2001) 50], the closed loop solitons in a plane, i.e., loops whose curvatures obey the modified Korteweg–de Vries equations, were investigated for the case related to algebraic curves with genera 1 and 2. This paper is a generalization of the previous paper to those of hyperelliptic curves with general genera. It was proved that the tangential angle of loop soliton is expressed by the Weierstrass hyperelliptic al-function for a given hyperelliptic curve $y^2 = f(x)$ with genus g.

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1. Introduction

This paper is on loop solitons related to hyperelliptic curves with higher genera as an extension of the previous report [23].

In [20], a problem of a quantized elastica (ideal thin elastic curve), or statistical mechanics of elasticas, was proposed, which is a model of a large polymer in a plane such as DNA at finite temperature. When a position of the elastica in a complex plane $\mathbb{C}$ is denoted by $Z : S^1 \hookrightarrow \mathbb{C}$, the partition function of elastica is given by

$$\mathcal{Z}[\beta] := \int DZ \exp(-\beta E[Z]), \tag{1.1}$$

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where β is the inverse of temperature, DZ a certain functional measure and $E[Z]$ the Euler–Bernoulli functional energy which is given by

$$E[Z] := \oint \mathrm{d}s\, k(s)^2 \tag{1.2}$$

for the arclength s, i.e., induced metric of the curve and its curvature k.

As shown in [20–22], the partition function is completely determined by the orbits of the modified Korteweg–de Vries (MKdV) hierarchical equations. As a curve obeying the MKdV equation is known as a loop soliton due to [12,16], the quantized elastica problem is a realization of the loop solitons. It is worthwhile noting that even though the soliton theory is studied in field of physics, there are not so many examples that soliton equation is connected with a physical model including its multi-soliton solutions.

In this paper, we will consider hyperelliptic solutions of the loop solitons or excited states of a quantized elastica. Theorem 3.2 is our main theorem of this paper. There we give explicit solutions of closed loop solitons in a plane related to hyperelliptic curves with general genera.

We will base on the result of [25]; there we show hyperelliptic solutions of the MKdV equation in terms of the theories of the hyperelliptic functions which were developed in the 19th century [2–4,9,14] and are recently re-evaluated [7,8,23,24]. Following the idea mentioned in the discussion in [23], we extend the results in [23] to the case of general genus in terms of Weierstrass's hyperelliptic al-function [3, p. 34; 31].

As the elastica problem has a deep history [9,17,29,30], I believe that one of its reasons is its naturalness. In the derivations of the solutions, it turns out that the elastica problem is very natural even from a mathematical viewpoint. In Remark 3.3, we will give comments on its naturalness.

Further as mentioned in [19], the elastica problem is closely related to automorphic function theory even though the solutions are constructed in abelian variety. In fact the Euler–Bernoulli energy functional can be expressed by the Schwarz derivative

$$E[Z] = \oint \mathrm{d}s \{Z, s\}_{\mathrm{SD}}. \tag{1.3}$$

In Section 4, we comment on its relation to automorphic functions.

2. Differentials of a hyperelliptic curve

In this section, we will review the hyperelliptic functions following [2–4,8,27] without explanations and proofs. We denote the set of complex number by $\mathbb{C}$ and the set of integers by $\mathbb{Z}$.

Convention 2.1. *We deal with a hyperelliptic curve X_g of genus g ($g > 0$) given by the affine equation*

$$y^2 = f(x) = \lambda_{2g+1}x^{2g+1} + \lambda_{2g}x^{2g} + \cdots + \lambda_2 x^2 + \lambda_1 x + \lambda_0 = P(x)Q(x), \tag{2.1}$$

where

$$\begin{aligned} f(x) &= (x-b_1)(x-b_2)\cdots(x-b_{2g})(x-b_{2g+1}),\\ Q(x) &= (x-c_1)(x-c_2)\cdots(x-c_g)(x-c),\\ P(x) &= (x-a_1)(x-a_2)\cdots(x-a_g), \end{aligned} \tag{2.2}$$

$\lambda_{2g+1} \equiv 1$, *and* λ_j*'s,* a_j*'s,* b_j*'s,* c_j*'s and* c *are complex values.*

Definition 2.2 ([2,3,7,8,27]). For a point $(x_i, y_i) \in X_g$, we define the following quantities:

(1) Let us denote the homology of a hyperelliptic curve X_g by

$$\mathrm{H}_1(X_g, \mathbb{Z}) = \bigoplus_{j=1}^{g} \mathbb{Z}\alpha_j \oplus \bigoplus_{j=1}^{g} \mathbb{Z}\beta_j, \tag{2.3}$$

where these intersections are given as $[\alpha_i, \alpha_j] = 0\,[\beta_i, \beta_j] = 0$ and $[\alpha_i, \beta_j] = \delta_{i,j}$.

(2) The unnormalized differentials of the first kind are defined by

$$\begin{aligned} du_1^{(i)} &:= \frac{dx_i}{2y},\\ du_2^{(i)} &:= \frac{x_i\, dx_i}{2y},\\ &\vdots\\ du_g^{(i)} &:= \frac{x_i^{g-1}\, dx_i}{2y}. \end{aligned} \tag{2.4}$$

(3) The unnormalized period matrices are defined by

$$\boldsymbol{\omega}' := \left[\left(\int_{\alpha_j} du_i^{(a)}\right)_{ij}\right], \quad \boldsymbol{\omega}'' := \left[\left(\int_{\beta_j} du_i^{(a)}\right)_{ij}\right], \qquad \boldsymbol{\omega} := \begin{bmatrix} \boldsymbol{\omega}' \\ \boldsymbol{\omega}'' \end{bmatrix}. \tag{2.5}$$

(4) The normalized period matrices are given by

$$\begin{aligned} &{}^t[\,\hat{\omega}_1 \quad \cdots \quad \hat{\omega}_g\,] := \boldsymbol{\omega}'^{-1\,t}[\,du_1^{(i)} \quad \cdots \quad du_g^{(i)}\,],\\ &\boldsymbol{\tau} := \boldsymbol{\omega}'^{-1}\boldsymbol{\omega}'', \qquad \hat{\boldsymbol{\omega}} := \begin{bmatrix} 1_g \\ \boldsymbol{\tau} \end{bmatrix}. \end{aligned} \tag{2.6}$$

(5) The unnormalized differentials of the second kind are defined by

$$\begin{aligned} d\tilde{u}_1^{(i)} &:= \frac{x_i^{g}\, dx_i}{2y_i},\\ d\tilde{u}_2^{(i)} &:= \frac{x_i^{g+1}\, dx_i}{2y_i},\\ &\vdots\\ d\tilde{u}_g^{(i)} &:= \frac{x_i^{2g-1}\, dx_i}{2y_i}, \end{aligned} \tag{2.7}$$

and $\mathrm{d}\mathbf{r}^{(i)} := (\mathrm{d}r_1^{(i)}, \mathrm{d}r_2^{(i)}, \dots, \mathrm{d}r_g^{(i)})$,

$$(\mathrm{d}\mathbf{r}^{(i)}) := \varLambda \begin{pmatrix} \mathrm{d}\mathbf{u}^{(i)} \\ \mathrm{d}\tilde{\mathbf{u}}^{(i)} \end{pmatrix}, \tag{2.8}$$

where $\varLambda$ is $2g \times g$ matrix defined by

$$\varLambda = \left(\begin{array}{cccccccccccc} 0 & \lambda_3 & 2\lambda_4 & 3\lambda_5 & \cdots & (g-1)\lambda_{g+1} & g\lambda_{g+2} & (g+1)\lambda_{g+3} & \cdots & (2g-3)\lambda_{2g-1} & (2g-2)\lambda_{2g} & (2g-1)\lambda_{2g+1} \\ & 0 & \lambda_5 & 2\lambda_6 & \cdots & (g-2)\lambda_{g+2} & (g-1)\lambda_{g+3} & g\lambda_{g+4} & \cdots & (2g-4)\lambda_{2g} & (2g-3)\lambda_{2g+1} & 0 \\ & & 0 & \lambda_7 & \cdots & (g-3)\lambda_{g+3} & (g-2)\lambda_{g+4} & (g-1)\lambda_{g+5} & \cdots & (2g-5)\lambda_{2g+1} & 0 & \\ & & & & \ddots & \vdots & \vdots & \vdots & \cdots & 0 & & \\ & & 0 & & & \lambda_{2g-2} & 2\lambda_{2g-1} & 3\lambda_{2g+1} & & & & \\ & & & & & 0 & \lambda_{2g+1} & 0 & & & 0 & \end{array} \right). \tag{2.9}$$

(6) The complete hyperelliptic integral matrices of the second kind are defined by

$$\boldsymbol{\eta}' := \left[\left(\int_{\alpha_j} \mathrm{d}r_i^{(a)} \right)_{ij} \right], \qquad \boldsymbol{\eta}'' := \left[\left(\int_{\beta_j} \mathrm{d}r_i^{(a)} \right)_{ij} \right], \qquad \boldsymbol{\omega} := \begin{bmatrix} \boldsymbol{\omega}' \\ \boldsymbol{\omega}'' \end{bmatrix}. \tag{2.10}$$

(7) By defining the Abel map for gth symmetric product of the curve X_g,

$$\mathbf{u} : \mathrm{Sym}^g(X_g) \to \mathbb{C}^g,$$
$$\left(u_k((Q_i)_{i=1,\dots,g}) := \sum_{i=1}^{g} \int_{\infty}^{Q_i} \mathrm{d}u_k^{(i)}, \quad k = 1, \dots, g \right), \tag{2.11}$$

the Jacobi variety $\mathcal{J}_g$ are defined as complex torus

$$\mathcal{J}_g := \frac{\mathbb{C}^g}{\mathbf{\Lambda}}. \tag{2.12}$$

Here $\mathbf{\Lambda}$ is a lattice generated by $\boldsymbol{\omega}$.

Definition 2.3. The coordinate in $\mathbb{C}^g$ for points $\{Q_i \equiv (x_i, y_i) | i = 1, \dots, g\}$ of the curve $y^2 = f(x)$ is given by

$$u_j := \sum_{i=1}^{g} \int_{\infty}^{(x_i, y_i)} \mathrm{d}u_j^{(i)}, \qquad \mathrm{d}u_j = \sum_{i=1}^{g} \mathrm{d}u_j^{(i)}. \tag{2.13}$$

(1) Using the coordinate u_j, σ functions, which is a holomorphic function over $\mathbb{C}^g$, is defined by [3, pp. 336 and 350; 7,14]

$$\sigma(u) = \sigma(u; X_g) := \gamma \exp\left(-\frac{1}{2} {}^t u \boldsymbol{\eta}' \boldsymbol{\omega}'^{-1} u \right) \vartheta \begin{bmatrix} \delta'' \\ \delta' \end{bmatrix} (\boldsymbol{\omega}'^{-1} u; \boldsymbol{\tau}), \tag{2.14}$$

where γ is a certain constant factor

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z;\boldsymbol{\tau}) := \sum_{n\in\mathbb{Z}^g} \exp\left[2\pi\sqrt{-1}\left\{\frac{1}{2}{}^t(n+a)\boldsymbol{\tau}(n+a)+{}^t(n+a)(z+b)\right\}\right] \tag{2.15}$$

for g-dimensional vectors a and b, and

$$\delta' := {}^t\begin{bmatrix} \frac{g}{2} & \frac{g-1}{2} & \cdots & \frac{1}{2} \end{bmatrix}, \qquad \delta'' := {}^t\begin{bmatrix} \frac{1}{2} & \cdots & \frac{1}{2} \end{bmatrix}. \tag{2.16}$$

(2) Hyperelliptic $\wp$-function is defined by [2,3,14]

$$\wp_{ij}(u) := -\frac{\partial^2}{\partial u_i \partial u_j}\log\sigma(u), \tag{2.17}$$

and hyperelliptic ζ_i function is defined by

$$\zeta_i(u) := \frac{\partial}{\partial u_j}\log\sigma(u). \tag{2.18}$$

(3) Weierstrass hyperelliptic al_r-function is defined by [3, p. 340; 31]

$$\mathrm{al}_r(u) := \gamma'\sqrt{F(b_r)}, \tag{2.19}$$

where γ' is a certain constant

$$F(x) := (x-x_1)\cdots(x-x_g) = \gamma_g x^g + \gamma_{g-1}x^{g-1} + \cdots + \gamma_0, \tag{2.20}$$

where $\gamma_g \equiv 1$

Proposition 2.4.

(1) $\wp_{gi}$ $(i=1,\ldots,g)$ *is elementary symmetric functions of* $\{x_1, x_2, \ldots, x_g\}$ [2–4,7], *i.e.*

$$F(x) = x^g - \sum_{i=1}^{g}\wp_{gi}x^{i-1}. \tag{2.21}$$

(2) $\zeta_j(u)$ *is expressed by* [8, pp. 33–35]

$$-\zeta_i(u) = \sum_{k=1}^{g}\int_{\infty}^{x_i} \mathrm{d}r_i - \frac{1}{2}\det A_{g-i}, \tag{2.22}$$

where

$$A_{n+1} := \begin{pmatrix} e_1 & -1 & & & & & \\ 2e_2 & -e_1 & 1 & & 0 & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ (n-2)e_{n-2} & -e_{n-3} & e_{n-4} & \cdots & \pm 1 & & \\ (n-1)e_{n-1} & -e_{n-2} & e_{n-3} & \cdots & \pm e_1 & \mp 1 & \\ ne_n & -e_{n-1} & e_{n-2} & \cdots & \pm e_2 & \mp e_1 & \pm 1 \\ (n+1)d_n & -d_{n-1} & d_{n-2} & \cdots & \pm d_2 & \mp d_1 & \pm 1 \end{pmatrix}, \tag{2.23}$$

and $e_i := \wp_{g,g+1-i}$ *and* $d_i := \wp_{g,g,g+1-i} := \partial\wp_{g,g+1-i}/\partial u_g$.

As we show later, (2.22) is a very important relation in our quantized elastica, which was found by Buchstaber et al. [8] (Appendix A); according to Buchstaber et al. [8], Baker [2] gave a wrong relation. We note that $F(x)$ is a generator of the elementary symmetric functions and the matrix A_n (2.23) contains the matrix of the Newton formula as its minor matrix [18]. In fact in the derivation of (2.23), A_n plays the role which connects the elementary and power sum symmetric functions.

Definition 2.5.

(1) A polynomial associated with $F(x)$ is introduced by

$$\pi_i(x) := \frac{F(x)}{x - x_i} = \chi_{i,g-1}x^{g-1} + \chi_{i,g-2}x^{g-2} + \cdots + \chi_{i,1} + \chi_{i,0}, \tag{2.24}$$

where $\chi_{i,g-1} \equiv 1$, $\chi_{i,g-2} = (x^1 + \cdots + x_g) - x_i$, and so on.

(2) We will introduce $g \times g$-matrices

$$W := \begin{pmatrix} \chi_{1,0} & \chi_{1,1} & \cdots & \chi_{1,g-1} \\ \chi_{2,0} & \chi_{2,1} & \cdots & \chi_{2,g-1} \\ \vdots & \vdots & \ddots & \vdots \\ \chi_{g,0} & \chi_{g,1} & \cdots & \chi_{g,g-1} \end{pmatrix}, \qquad \mathcal{Y} := \begin{pmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_g \end{pmatrix},$$

$$\mathcal{F}' := \begin{pmatrix} F'(x_1) & & & \\ & F'(x_2) & & \\ & & \ddots & \\ & & & F'(x_g) \end{pmatrix}, \tag{2.25}$$

where $F'(x) := \mathrm{d}F(x)/\mathrm{d}x$.

(3)

$$M := \begin{pmatrix} 1 & & & & \\ \gamma_{g-1} & 1 & & 0 & \\ \gamma_{g-2} & \gamma_{g-1} & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{g-1} & 1 \end{pmatrix}, \quad K := \begin{pmatrix} x_1^{g-1} & x_1^{g-2} & \cdots & 1 \\ x_2^{g-1} & x_2^{g-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_g^{g-1} & x_g^{g-2} & \cdots & 1 \end{pmatrix}. \tag{2.26}$$

(4) The coordinate in $\mathbb{C}^g$ is introduced by $\mathbf{u}^{(r)} := \mathcal{P}_r\mathbf{u}$, where $\mathcal{P}_r$ is defined by its inverse matrix

$$\mathcal{P}_r^{-1} := \begin{pmatrix} 1 & gb_r & \begin{pmatrix} g-1 \\ 2 \end{pmatrix} b_r^2 & \cdots & \begin{pmatrix} g-1 \\ g-1 \end{pmatrix} b_r^{g-1} & b_r^{g-1} \\ 0 & 1 & (g-1)b_r & \cdots & \begin{pmatrix} g-1 \\ g-2 \end{pmatrix} b_r^{g-3} & b_r^{g-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & b_r \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{2.27}$$

(5) For a polynomial $g(X) = g_n X^n + \cdots + g_0$, we introduce the D_j operator

$$D_j = \sum_{i=j}^{n} g_i X^{i-j}. \tag{2.28}$$

Lemma 2.6.

(1) *The inverse matrix of W is given by* $W^{-1} = \mathcal{F}^{-1}V$, *where V is Vandermonde matrix*

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_g \\ x_1^2 & x_2^2 & \cdots & x_g^2 \\ \vdots & \vdots & & \vdots \\ x_1^{g-1} & x_2^{g-1} & \cdots & x_g^{g-1} \end{pmatrix}. \tag{2.29}$$

(2) *Let* $\partial_{u_i} := \partial/\partial u_i$, $\partial_{x_i} := \partial/\partial x_i$ *and* $\partial_{u_i}^{(r)} := \partial/\partial u_i^{(r)}$,

$$\begin{pmatrix} \partial_{u_1} \\ \partial_{u_2} \\ \vdots \\ \partial_{u_g} \end{pmatrix} = 2\mathcal{Y}\mathcal{F}'^{-1}W \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \vdots \\ \partial_{x_g} \end{pmatrix}, \qquad \begin{pmatrix} \partial_{u_1}^{(r)} \\ \partial_{u_2}^{(r)} \\ \vdots \\ \partial_{u_g}^{(r)} \end{pmatrix} = {}^t\mathcal{P}^{-1} \begin{pmatrix} \partial_{u_1} \\ \partial_{u_2} \\ \vdots \\ \partial_{u_g} \end{pmatrix}. \tag{2.30}$$

(3) *$KM = W$ and*

$$W_{ij} = \chi_{i,j-1} = [D_j(F(X))]_{X=x_i}. \tag{2.31}$$

Proof. Case (1) is obvious by using the properties of the Vandermonde matrix. In case (2), we must pay attention the fixed parameters for the partial differential. By comparing $\mathrm{d}x^i$ and the chain relation of ∂_{u_i}, we obtain the matrix representation (2.30) [25]. From the relation $(F(x)/(x - x_i))(x - x_i) = F(x)$, we have

$$\chi_{i,j} = \gamma_{j+1} + x_i\chi_{i,j+1}. \tag{2.32}$$

Then we obtain the relation (2.31). □

We note that formulae (2.31) and (2.32) are very important to prove (2.23).

Proposition 2.7 ([8, p. 11]). *The Legendre relation is given by*

$${}^t\omega'\eta'' - {}^t\omega''\eta' = 2\pi\sqrt{-1}I_g, \tag{2.33}$$

where I_g is the $g \times g$-unit matrix.

3. Loop solitons

In this section, we will deal with a real curve in a plane in the category of differential geometry.

Let us consider a smooth immersion of a circle S^1 into the two-dimensional Euclidean space $\mathbb{E}^2 \approx \mathbb{C}$ or $\mathbb{E}^2 + \{\infty\} \approx \mathbb{C}P^1$. The immersed real curve C is characterized by the affine coordinate $(X^1(s), X^2(s))$ around the origin. Here s is a parameter of S^1 and is, now, chosen as the arclength so that $\mathrm{d}s^2 = (\mathrm{d}X^1)^2 + (\mathrm{d}X^2)^2$. We will also use the complex expression

$$Z(s) := X^1(s) + \sqrt{-1}X^2(s). \tag{3.1}$$

Then by letting $\partial_s := \partial/\partial s$, $|\partial_s Z(s)| = 1$ and the curvature of C is given by

$$k(s) := \frac{1}{\sqrt{-1}}\partial_s \log \partial_s Z(s). \tag{3.2}$$

As mentioned in Section 1, loop soliton is identified with a quantized elastica [20]. Thus we will sometimes call it *quantized elastica* or simply *elastica* hereafter.

Definition 3.1.

(1) A one-parameter family of curves $\{C_t\}$ for a real parameter $t \in \mathbb{R}$ is called a loop soliton, if its curvature obeys the MKdV equation, for $q := k/2$,

$$\partial_t q + 6q^2\partial_s q + \partial_s^3 q = 0, \tag{3.3}$$

where $\partial_t := \partial/\partial t$.

(2) The energy of elastica is given by

$$E[Z] := \oint \mathrm{d}s\, k^2, \tag{3.4}$$

which can be expressed by the Schwarz derivative

$$E[Z] = \oint \mathrm{d}s \{Z, s\}_{\mathrm{SD}}, \tag{3.5}$$

where

$$\{Z, s\}_{\mathrm{SD}} := \partial_s \left(\frac{\partial_s^2 Z}{\partial_s Z} \right) - \frac{1}{2} \left(\frac{\partial_s^2 Z}{\partial_s Z} \right)^2. \tag{3.6}$$

Here we will give our main theorem as follows.

Theorem 3.2. *Let the configuration of the x-components* $(x_1, \dots, x_g)$ *of the affine coordinates of the hyperelliptic curves* $\mathrm{Sym}^g(X_g)$ *satisfy*

$$|F(b_r)| = r_0, \tag{3.7}$$

where r_0 *is a positive number. For such* $(x_1, y_1), \dots, (x_g, y_g)$, *we have* $\mathbf{u} := \mathbf{u}((x_1, y_1), \dots, (x_g, y_g))$ *due to* (2.11).

(1) *By setting* $s \equiv u_g/r_0$ *and* $t \equiv u_{g-1} + (\lambda_{2g-1} + b_r)u_g$,

$$\partial_{u_g} Z^{(r)} := F(b_r) \quad \text{or} \quad |\partial_s Z^{(r)}| = 1, \tag{3.8}$$

completely characterizes the loop soliton.

(2) *The shape of elastica is given by*

$$Z^{(r)} = \frac{1}{r_0} \left(b_r^g u_g + \sum_{i=1}^{g} b_r^i \zeta_{i-1} \right). \tag{3.9}$$

Remark 3.3.

(1) If one prefers more proper expression for the branch point $(b_r, 0)$, he may use $u_j^{(r)}$ in (2.27) and then find similar results. As the expression is essentially the same as the above one due to (2.30), we will investigate the above one.

(2) The condition (3.7) is essential. Due to the condition, any configurations of $\mathbf{u}$, the tangential angle $\phi = \log(\partial_{u_g} Z)/\sqrt{-1}$ does not contain imaginary part. Hence the arclength locally does not change [19–22]. As Goldstein and Petrich [11] showed that the isometric deformation of space curve in a plane gives the MKdV equation, this condition and $[\partial_{u_g}, \partial_{u_i}] = 0$ recover the MKdV equation in general.

(3) The tangential vector, $\partial_{u_g} Z \equiv F(b_r) \propto \mathrm{al}_r^2$, consists only of x_i's, which can be regarded as a twofold coordinate of $\mathrm{Sym}^g(\mathbb{C}P^1)$. Each $x_i \in \mathbb{C}P^1$ appears when we construct the hyperelliptic curve X_g using two $\mathbb{C}P^1$ with $g + 1$ cuts.

(4) Due to the configuration of (3.7), there is a trivial action of the $U(1)$-group, which exhibits the translation symmetry of the elastica. When we begin with this symmetry and isometric deformation, we reproduces the MKdV hierarchy [20,21].
(5) The condition (3.7) should be regarded as a reality condition. Elastica problem is a real analytic problem. In the primitive sense, the complex analysis is more complex than the real analysis but from deeper viewpoint, their standpoints are reversed. In fact, due to the condition, we must investigate all possible contours in the complex curve X_g. In other words, from the point of view of real analysis, as long as we have insufficient knowledge of the condition, it is not the end of the study of the quantized elastica problem. I suppose that this difficulty is similar to that of real analytic Eisenstein series [28].
(6) The condition (3.7) is satisfied if all x_i's are in a circle centralizing at b_r. Then symmetric configuration of $x_1, \dots, x_g$ determines a point of a shape of the loop soliton. In other words, the dynamics of the elastica is translated to symmetric system of g-particles in S^1. Dynamics of symmetric particles in a circle might be familiar with researches of quantum integrable system [15].

When we consider the discrete configurations of x's, they give the discrete time development of the piecewise linear curves. This must be related to the discrete integrable system.

From the definition, Theorem 3.2 can be proved by the following proposition, which was shown in [25]. We will give a sketch of the proof of [25], whose techniques essentially appeared in [4].

Proposition 3.4. *By letting*

$$\mu^{(r)} := \frac{1}{2}\partial_{u_g}\phi^{(r)}, \quad \phi^{(r)}(u) := \frac{1}{\sqrt{-1}}\log F(b_r), \tag{3.10}$$

$\mu^{(r)}$ *obeys the modified KdV equation*

$$(\partial_{u_{g-1}} - (\lambda_{2g} + b_r)\partial_{u_g})\mu^{(r)} - 6\mu^{(r)^2}\partial_{u_g}\mu^{(r)} + \partial^3_{u_g}\mu^{(r)} = 0. \tag{3.11}$$

Proof. This is proved in [20]. We will give a sketch of the proof. From the definition, we have

$$\frac{\partial}{\partial u_g}\log F(b_r) = \sum_{i=1}^{g} \frac{2y_i}{F'(x_i)(x_i - b_r)},$$
$$\frac{\partial}{\partial u_{g-1}}\log F(b_r) = \sum_{i=1}^{g} \frac{2y_i \chi_{i,g-1}}{F'(x_i)(x_i - b_r)}. \tag{3.12}$$

Let ∂X_g^0 is boundary when X_g is embedded in a upper-half plane $\mathcal{H}$. By estimation of

$$\oint_{\partial X_g^0} \frac{f(x)}{(x-b_r)F(x)^2}\,\mathrm{d}x = 0, \tag{3.13}$$

and counting its residues, we obtain

$$\sum_{k=1}^{g} \frac{1}{F'(x_k)} \left[\frac{\partial}{\partial x} \left(\frac{f(x)}{(x-b_r)F^2(x)} \right) \right]_{x=x_k} = \lambda_{2g} + b_r + 2\wp_{gg}. \tag{3.14}$$

Further, we have

$$\left(\sum_k \frac{y_k}{(x-x_k)F'(x_k)} \right)^2 = \sum_{k,l,k\not\equiv l} \frac{2y_k y_l}{(x-x_k)(x_k-x_l)F'(x_k)F'(x_l)} + \sum_k \frac{y_k^2}{(x-x_k)^2 F'(x_k)^2}. \tag{3.15}$$

Using them, we have the relation. □

We note that the formal power series

$$\mu^{(r)} \equiv \frac{1}{2\sqrt{-1}} \frac{\partial}{\partial u_g} \log F(b_r) = \frac{1}{\sqrt{-1}} \sum_{i=1}^{g} \frac{y_i}{F'(x_i)(x_i - b_r)} = \frac{1}{\sqrt{-1}} \sum_{j=1}^{\infty} \sum_{i=1}^{g} \frac{y_i}{F'(x_i) b_r} \frac{x_i^j}{b_r^j} \tag{3.16}$$

is resemble to the generator of the power sum symmetric functions [18].

As we proved Proposition 3.4, we give two corollaries, which are shown by direct computations. Corollary 3.5 gives local properties of the elastica and Corollary 3.6 is associated with its global properties.

Corollary 3.5.

(1) *The shape of elastica is given by*

$$Z^{(r)} = \frac{1}{r_0} \left(b_r^g u_g + \sum_{i,j=1}^{g} b_r^i \int^{(x_j,y_j)} \mathrm{d}r_{i-1} + \frac{1}{2} \sum_{i,j=1}^{g} b_r^i \det A_{g-i} \right). \tag{3.17}$$

(2) *The Schwarz derivative of Z with respect to* u_g,

$$\{Z^{(r)}, u_g\}_{\mathrm{SD}} = 4\wp_{gg} + 2\lambda_{2g} + 2b_r. \tag{3.18}$$

(3) *The root square of the tangential vector* $\sqrt{\partial_{u_g} Z^{(r)}} \equiv \mathrm{al}_r/\gamma'$ *is a solution of the Dirac equation or Frenet–Serret equation* [19]

$$\begin{pmatrix} \partial_{u_g} & \mu^{(r)} \\ \mu^{(r)} & -\partial_{u_g} \end{pmatrix} \begin{pmatrix} \sqrt{\partial_{u_g} Z^{(r)}} \\ \sqrt{-\partial_{u_g} Z^{(r)}} \end{pmatrix} = 0. \tag{3.19}$$

Here we comment on (3.18) and (3.19). First we note that the solution of the Dirac equation consists of al-functions due to (2.19) and (3.8). Second it is noted that from the definitions (3.6) and (3.10), and (3.18) agrees with the Miura transformation

$$\mu^{(r)2} + \sqrt{-1}\partial_{u_g}\mu^{(r)} = 2\wp_{gg} + \lambda_{2g} + b_r, \tag{3.20}$$

because the left-hand side consists of the solutions of the MKdV equation (3.11) whereas the right-hand side obeys the KdV equation [7,24]. Further it is obvious that (3.19) has the same data as the Miura transformation (3.20) by operating the Dirac operator twice [25]. On the other hand, (3.19) can be expressed by

$$\begin{pmatrix} -\partial_{u_g} & 0 \\ 0 & \partial_{u_g} \end{pmatrix} \begin{pmatrix} \sqrt{\partial_{u_g} Z^{(r)}} \\ \sqrt{-\partial_{u_g} Z^{(r)}} \end{pmatrix} = \begin{pmatrix} 0 & \mu^{(r)} \\ \mu^{(r)} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\partial_{u_g} Z^{(r)}} \\ \sqrt{-\partial_{u_g} Z^{(r)}} \end{pmatrix}. \tag{3.21}$$

Here we can recognize that the left-hand side is an operation in analytic category while right-hand side is an operation as an endmorphisms in a commutative algebra. This relation is essential in the study of $\mathcal{D}$-module, due to the statements in [6, pp. 12–13].

Corollary 3.6.

(1) *The winding number of elastica can be computed for a given path by the integration*

$$w := \frac{1}{2\pi}\oint \partial_{u_g}\phi(u_g)\,\mathrm{d}u_g. \tag{3.22}$$

(2) *The closed condition of elastica*

$$\oint \partial_{u_g} Z^{(r)}\,\mathrm{d}u_g \equiv 0 \tag{3.23}$$

consists of the conditions

$$b^{(r)^g}\omega_i + \sum_{i=1}^{g} b^{(r)^i}\eta_{i-1} = 0, \tag{3.24}$$

using the hyperelliptic integral (2.5) and (2.6).

(3) $$\{Z^{(r)}, u_g\}_{\mathrm{SD}}\,\mathrm{d}u_g = -4\,\mathrm{d}\zeta_g + 2(\lambda_{2g} + b_r)\,\mathrm{d}u_g. \tag{3.25}$$

(4) $$\oint_{\beta_a}\{Z^{(r)}, u_g\}_{\mathrm{SD}}\,\mathrm{d}u_g = -4\eta''_{ag} + 2(\lambda_{2g} + b_r)\omega''_{ag}, \tag{3.26}$$

$$\oint_{\alpha_a}\{Z^{(r)}, u_g\}_{\mathrm{SD}}\,\mathrm{d}u_g = -4\eta'_{ag} + 2(\lambda_{2g} + b_r)\omega'_{ag}. \tag{3.27}$$

4. Discussion

As mentioned in [19], our problem is resemble to Poincaré, Klein and Schwarz theories of the automorphic function over the half plane or the Poincaré disk.

First the resemblance is obtained through the conformal field theory. Since S^1 is homotopically equivalent with $\mathbb{C} - \{0\}$, most results of the conformal field theory are obtained by the investigation of dynamics of functions over S^1 in a Riemann sphere $\mathbb{C}P^1$ [13]. The situation in (3.7) is very resemble to that of the conformal field theory but is in the higher genus Riemannian surface. However, we can also perform the Fourier transformation or localization around the ramified point $(b_r, 0)$. Then it should be noted that on the injective maps

$$\partial_s Z^{(r)} : S^1 \to X_g, \tag{4.1}$$

there are nontrivial actions of the fundamental group (3.23). By letting

$$L_n := \oint (2\wp_{gg} + \lambda_{2g} + b_r)\, \mathrm{e}^{2\pi\sqrt{-1}nu_g/r_0} \frac{\mathrm{d}u_g}{r_0}, \tag{4.2}$$

we have the relation of Virasoro algebra [5,19]

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{0,n+m} n(n^2-1). \tag{4.3}$$

Here we remark that in the conformal field theory, the surface obtained by means of the Vertex operator acting a Riemannian sphere is called a Riemannian surface with genus g but as in showed in [24], such a Riemannian surface, at least of the case of the hyperelliptic curve, is very far from our Riemannian surfaces. The Riemannian surface obtained by the Vertex operator from the Riemannian sphere is, in fact, topologically genus g surface but is very special (semi-stable) degenerate curve; the theory over such a curve should be regarded as a theory on Riemannian sphere [1]. Recently some of the physicists might regard that topological aspect in field theory is the most important and they deal only with degenerate curves. However, at least, in low-energy physics related to our lives, we need finer topology. In fact, the shape of classical elastica, which was studied by Euler as a classical field theory and is determined by curvature, leads us to very fruitful physics and mathematics. Further in general, physical phenomena are not in a framework of complex analysis. Even though some objects in the category of the complex analysis is classified by topological objects, it is not all in physics. For example, Euler equation for complete fluid dynamics in three-dimensional space cannot be expressed by complex analysis. Quantized elastica problem should also be considered in the real analytic category as Euler did. Of course, topological aspect is still important but, I believe, is not a goal for quantitative science.

Next I will comment on interesting relations, which also looks connected with the automorphic functions. $Z^{(r)}$ is roughly equal to ζ functions due to (3.9) whereas the incomplete "energy integral",

$$\int^{\mathbf{u}} \{Z^{(r)}, u_g\}_{\mathrm{SD}}\, \mathrm{d}u_g, \tag{4.4}$$

is also expressed by the ζ functions. It implies that there is a similarity between the configuration and energy of elasticas. Further the matrix (2.23) is essentially the same as the Newton formula which connects the elementary and power sum symmetric functions. Thus ζ (the configurations and energy from above view points) is essentially expressed by the power sum symmetric function, while the main part of μ, or a half curvature of elastica, is the same as a generator of the power sum symmetric function due to (3.16) [18]. It is expected that there might be hidden symmetries. These facts might remind us of replicability of automorphic function if they were related to automorphic functions [26]. In fact, for the case of elliptic function ($g = 1$) case helping with the notations in [23] ($(\partial_u \wp(u))^2 = 4\wp^3 - g_2\wp + g_3 = 4(\wp - e_1)(\wp - e_3)(\wp - e_4)$, $\wp(\omega_i) = e_i$), we have interesting formulae

$$\partial_u Z^{(a)}(u + \omega_a) = \tfrac{1}{4}\{Z^{(a)}(u), u\}_{\mathrm{SD}} - \tfrac{3}{2}e_1, \tag{4.5}$$

$$Z^{(a)}(u) = \lim_{\epsilon\to 0}\int^u \mathrm{d}u \frac{1}{\sigma(\epsilon)^2} \times \exp\left(-\frac{1}{2}\int_\epsilon^u \int_0^{u'} [\{Z^{(a)}(u''), u''\}_{\mathrm{SD}} - \{Z^{(a)}(u'' - \omega_a), u''\}_{\mathrm{SD}}]\,\mathrm{d}u''\,\mathrm{d}u'\right) \tag{4.6}$$

for $a = 1, 2, 3$. In the integrations, we should note the effects from the initial points. Eqs. (4.5) and (4.6) can be extended to higher genus. For the case of (4.6), we can do by using the relation al_r functions and σ function [3].

Further from the point of view of theory of the symmetric function, one might wish to regard x_i as an eigenvalue of some matrix, $\mathcal{X} = \mathrm{diag}(x_1, x_2, \dots, x_g)$. The generator of the elementary symmetric function is expressed by

$$\tilde{F}(x) = \det(\mathcal{X} - xI), \tag{4.7}$$

and that of the power symmetric function is expressed by

$$\tilde{G}(x) = \sum_{n=1}^{\infty} \mathrm{tr}(\mathcal{X}^n)x^n. \tag{4.8}$$

They are closely related to (2.20) and (3.16). As comparison with physics, some purposes in mathematics are classifications and determination of the relations among classified objects. The classification should be characterized by discrete quantities and these discrete quantities should sometimes preserve when we take a certain limit. Thus it might be natural to consider degenerate curves in a certain sense. The degenerate curve $y^2 = P(x)^2 x$, which was dealt with in [24] and is associated with the soliton solutions and algebra of vertex operators, is also expressed by $y^2 - x(\det(xI - \mathcal{P}))^2 = 0$ by letting $\mathcal{P} := \mathrm{diag}(a_1, a_2, \dots, a_g)$. These matrices $\mathcal{P}$ and $\mathcal{X}$ for the degenerate curve are the same rank. Hence it is natural that one consider a graded algebra $\mathcal{A}$ ($\mathcal{A}_g \subset \mathcal{A}_{g+1}$, $\mathcal{A} = \cup_g \mathcal{A}_g$) generated by generators ($\bar{\mathcal{X}}$, $\bar{\mathcal{P}}$ and so on) such that there is a $\mathcal{A}$-module M_g satisfying $\bar{\mathcal{X}} M_g = \mathcal{X} M_g$ and $\bar{\mathcal{P}} M_g = \mathcal{P} M_g$. If we regard M_g as a representation of shape of elastica, we might able to deal with family of elastica related to hyperelliptic curves with different genera. The Schwarz derivative, which plays important roles in theory of automorphic functions [26] and is invariant for $\mathrm{PSL}(2,\mathbb{C})$, naturally appears in our elastica problem [22]. Even though $\mathrm{PSL}(2,\mathbb{Z})$ is far from our situation in this stage, I hope that our study might reveal some relations between elastica problems and automorphic function theory [26], if exists [19].

Finally, we mention our future study. The quantized elastica in a plane was extended to that in $\mathbb{R}^3$. There the nonlinear Schrödinger equation and complex MKdV equation play the same role as the MKdV equation [21]. As the explicit function form of the finite type solution, the nonlinear Schrödinger equation was obtained in [10], this study might be extended to that in $\mathbb{R}^3$.

5. Note added in proof

Recently I learned that in [32] Mumford investigated the (classical) elastica problem of genus one from view point of applied mathematics and gave simple and deep expression of the shape of elastica, which is related to this study and [23].

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Appendix A

As (2.22) was given only in improved version [8] of published version [7], we briefly review its derivation here following [8]. There Buchstaber et al. started from a fundamental relation among the hyperelliptic sigma functions and incomplete integrals of the first, second and third kinds. Then they used the operator (2.28), investigated of symmetry of an intermediate equation and reached (2.22).

Thus let us follow their way. First we will give the fundamental formula in the σ function theory [2,7]

$$\log\left(\frac{\sigma\left(\int_\infty^{\mathrm{P}} \mathrm{d}\mathbf{u}+\mathbf{u}\right)\sigma\left(\int_\infty^{\mathrm{Q}} \mathrm{d}\mathbf{u}+\mathbf{u}'\right)}{\sigma\left(\int_\infty^{\mathrm{P}} \mathrm{d}\mathbf{u}+\mathbf{u}'\right)\sigma\left(\int_\infty^{\mathrm{Q}} \mathrm{d}\mathbf{u}+\mathbf{u}\right)}\right)=\sum_{j=1}^{g}\mathbf{R}^{\mathrm{P},\mathrm{Q}}_{\overline{\mathrm{P}_j},\overline{\mathrm{Q}_j}}, \tag{A.1}$$

where

$$\mathbf{u}=\sum_{j=1}^{g}\int_\infty^{\mathrm{P}_j}\mathrm{d}\mathbf{u}, \qquad \mathbf{u}'=\sum_{j=1}^{g}\int_\infty^{\mathrm{Q}_j}\mathrm{d}\mathbf{u}, \tag{A.2}$$

$$\mathbf{R}^{\mathrm{P},\mathrm{Q}}_{\mathrm{P}',\mathrm{Q}'}\equiv\mathbf{R}^{\mathrm{P}',\mathrm{Q}'}_{\mathrm{P},\mathrm{Q}}=\int_{\mathrm{Q}}^{\mathrm{P}}\mathrm{d}u_1\int_{\mathrm{Q}'}^{\mathrm{P}'}\mathrm{d}r_1+\cdots+\int_{\mathrm{Q}}^{\mathrm{P}}\mathrm{d}u_g\int_{\mathrm{Q}'}^{\mathrm{P}'}\mathrm{d}r_g+\mathbf{P}^{\mathrm{P},\mathrm{A}}_{\mathrm{Q},\mathrm{B}}, \tag{A.3}$$

$$\mathbf{P}^{\mathrm{P},\mathrm{Q}}_{\mathrm{P}',\mathrm{Q}'}:=\int_{\mathrm{Q}}^{\mathrm{P}}\Omega(\mathrm{P}',\mathrm{Q}'), \qquad \Omega(\mathrm{P},\mathrm{Q}):=\left(\frac{y+y_{\mathrm{P}}}{x-x_{\mathrm{P}}}-\frac{y+y_{\mathrm{Q}}}{x-x_{\mathrm{Q}}}\right)\frac{\mathrm{d}x}{2y}, \tag{A.4}$$

and $\overline{\mathrm{P}_j}$ $(\overline{\mathrm{Q}_j})$ is conjugate of P_j (Q_j) with respect to the symmetry of hyperelliptic curve $(x, y) \to (x, -y)$.

After letting all Q_j to ∞, we derivative it in u_j, we obtain

$$\zeta_j\left(\int_\infty^{\mathrm{P}} \mathrm{d}\mathbf{u} + \mathbf{u}\right) - \zeta_j\left(\int_\infty^{\mathrm{Q}} \mathrm{d}\mathbf{u} + \mathbf{u}\right) + \int_{\mathrm{Q}}^{\mathrm{P}} \mathrm{d}r_j - \frac{\partial}{\partial u_j}\sum_{i=1}^{g}\int_\infty^{\overline{\mathrm{P}}_i} \Omega(\mathrm{P}, \mathrm{Q}) = 0. \tag{A.5}$$

Introducing $R(z) = (z - x_0)F(z)$ for $\mathrm{P} = (x_0, y_0)$ and $\mathrm{P}_j = (x_j, y_j)$, (A.5) becomes

$$\begin{aligned} &\zeta_j\left(\int_\infty^{\mathrm{P}} \mathrm{d}\mathbf{u} + \mathbf{u}\right) + \int_\infty^{\mathrm{P}} \mathrm{d}r_j + \sum_{i=0}^{g}\int_\infty^{\mathrm{P}_j} \mathrm{d}r_j \\ &\quad - \frac{1}{2}\sum_{i=0}^{g} y_i\left(\left.\frac{D_j(R'(z)) - jD_{j+1}(R(z))}{R'(z)}\right|_{z=x_i}\right) \\ &= \zeta_j(\mathbf{u}) + \sum_{i=1}^{g}\int_\infty^{\mathrm{P}_j} \mathrm{d}r_j - \frac{1}{2}\sum_{k=1}^{g}\frac{1}{y_k}\frac{\partial x_k}{\partial u_j}\frac{1}{y_i}\frac{y_i - y_\infty}{x_i - x_\infty} \\ &\quad - \frac{1}{2}\sum_{i=1}^{g} y_i\left(\left.\frac{D_j(F'(z)) - jD_{j+1}(F(z))}{R'(z)}\right|_{z=x_i}\right). \end{aligned} \tag{A.6}$$

Then noting the fact that the left-hand side is symmetrical in $x_0, x_1, \ldots, x_g$, while the right-hand side is symmetrical in $x_1, x_2, \ldots, x_g$ but does not depend on x_0, (A.6) is reduced to (2.22). (2.23) comes from the third term in the left-hand side of (A.6).

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